Noncommutative radial waves

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Abstract

We study radial waves in (2+1)-dimensional noncommutative scalar field theory, using operatorial methods. The waves propagate along a discrete radial coordinate and are described by finite series deformations of Bessel-type functions. At large radius with respect to the noncomutativity scale θ, the waves behave like the usual commutative ones.

Field theory defined over a noncommutative space represents an interesting, nonlocal but most probably consistent, deformation of usual field theories. It also arises as the effective theory for the massless modes of string theory in a large B-field (see [1]). Noncommutative field theory displays an intriguing IR/UV mixing (see [2] and later works), demonstrated perturbatively but expected to hold in general. At the classical level, noncommutative theories possess solitonic solutions [3] which have no obvious counterpart in local field theory. Some recent discussions of the dynamics of these solutions can be found in [4]. However, in spite of considerable progress, noncommutative field theories are far from being well understood, even classically. One interesting issue, which might shed further light on these theories, is the description of oscillation and propagation processes on fuzzy spaces. A simple example will be discussed here.

The aim of this note is to describe radial waves in a $(2+1)$-dimensional (space-)noncomutative free scalar field theory. In contrast to plane waves, radial waves are affected by the presence of noncommutativity. They propagate on a discrete space, provided by the eigenvalues of the radius square

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operator. Their amplitude will be shown to be given by a finite series, which solves a discrete wave equation; the series is reminiscent of Bessel-type functions. In the large radius limit (corresponding to the 'large quantum number limit' in quantum mechanics) the number of terms of the series grows indefinitely. Then, the waves behave like the usual, commutative, radial ones, being described by the asymptotics of cylindrical functions. The nature of the solutions in the presence of sources will be touched upon at the end of this note.

We are going to use operatorial methods, which are quite straightforward in the present context. The Weyl-Moyal approach would require an unambiguous way to switch between Cartesian and polar noncommutative coordinates.

We start from the following action, written in operator form,

$$S = \int \mathrm{d}t \, \mathrm{Tr}_{\mathcal{H}} \left(\frac{1}{2} (\dot{\Phi}^2 + \frac{1}{2} [X_i, \Phi]^2 \right), \tag{1}$$

where $i = 1, 2$. The scalar field Φ is a time-dependent operator acting on the Hilbert space $\mathcal{H}$ on which the algebra

$$[x^1, x^2] = i\theta \tag{2}$$

is represented. There is no potential term, $V(\Phi) = 0$, since we study free waves. X_μ is given by $X_\mu = p_\mu + A_\mu$, where $p_i = \theta^{-1} \epsilon_{ij} x^j$. In the following the gauge field A_μ is taken to be zero; consequently we dropped further parts of the action which depend on it.

The equations of motion for the field Φ are

$$\ddot{\Phi} + [X_{\mu,} [X_\mu, \Phi]] = \ddot{\Phi} + \frac{1}{\theta^2} [x_i, [x_i, \Phi]] = 0. \tag{3}$$

In Cartesian coordinates, the solution of (3) is straightforward,

$$\Phi \sim e^{i(k_1 x_1 + k_2 x_2) - i\omega t}, \quad k_1^2 + k_2^2 = \omega^2, \tag{4}$$

and it describes plane waves, which are identical to the commutative ones. Since in a plane wave one drops the dependence upon the direction orthogonal to the wave vector, solution (4) is consistent with equation (2).

The novelty appears when one considers polar coordinates. If one chooses the oscillator basis $\{|n\rangle\}$ given by

$$N |n\rangle = n |n\rangle, \quad N = \bar{a} a, \quad a = \frac{1}{\sqrt{2\theta}} (x^1 + i x^2), \tag{5}$$

the equations of motion become

$$\ddot{\Phi} + \frac{2}{\theta}[a, [\bar{a}, \Phi]] = 0. \tag{6}$$

$N = \frac{1}{2}(\frac{x_1^2+x_2^2}{\theta} - 1)$ is basically the radius square operator, in units of θ. If one diagonalizes Φ in the $|n\rangle$ basis, its components are time-dependent c-numbers, $\Phi_n(t)$. They satisfy the equation

$$\ddot{\Phi}_n + \frac{2}{\theta}(n\Delta^2\Phi_{n-1} + \Delta\Phi_n) = 0, \quad n = 0, 1, 2, \ldots \tag{7}$$

in which the discrete derivative operator Δ is defined by

$$\Delta(\Phi_n) = \Phi_{n+1} - \Phi_n. \tag{8}$$

If one assumes the time dependence of Φ_n to be of the form $e^{i\omega t}$, one gets the difference equation

$$n\Delta^2\Phi_{n-1} + \Delta\Phi_n + \lambda\Phi_n = 0, \tag{9}$$

where $\lambda = \theta\omega^2/2$ for a massless scalar field. ($2\lambda/\theta = \omega^2 - m^2$ for a massive field, and $2\lambda/\theta = \omega^2 + m^2$ for a tachyon.)

Equation (9) has two independent linear solutions. We will find them in the form of (eventually finite) power series. First note that the standard power $n^k = n \cdot n \cdot \ldots \cdot n$ does not behave simply under the action of Δ. In order to adapt the usual logic of power series solutions to the above discrete equation, we define a different 'power of n', for n a positive integer:

$$n^{(k)} = n(n-1)(n-2)\ldots(n-k+1) = \frac{n!}{(n-k)!}. \tag{10}$$

It has the useful property that $\Delta n^{(k)} = kn^{(k-1)}$.

We now try to find a solution $\Phi(n, \sigma)$ of the form

$$\Phi(n, \sigma) = \sum_{k=0}^{\infty} a_k(\sigma) n^{(k+\sigma)}. \tag{11}$$

σ is an arbitrary parameter, to be fixed by the equation. Substituting (11) into (9), we obtain a recurrence relation for the coefficients $a_k(\sigma)$,

$$a_k(\sigma) = \frac{(-\lambda)}{(k+\sigma)^2} a_{k-1}(\sigma) = \frac{(-\lambda)^k}{(k+\sigma)^2(k-1+\sigma)^2\ldots(1+\sigma)^2} a_0 \tag{12}$$

and a condition for σ,

$$\sigma^2 = 0, \quad \text{i.e.} \quad \sigma_1 = \sigma_2 = 0. \tag{13}$$

Thus a first solution of our equation is

$$\Phi_1(n) = \sum_{k=0}^{n} \frac{(-\lambda)^k}{(k!)^2} n^{(k)}. \tag{14}$$

It is given by a *finite* sum, since $n^{(p>n)} = 0$ for n, p positive integers. It is understood that, if one calculates a discrete derivative of (i.e. apply Δ to) the above solution, one should put $\sigma = 0$ only after operating with Δ. An arbitrary dimensionfull constant, which should multiply the RHS of (14), will not be displayed; it can be reinstated at any moment.

Since equation (13) has two equal roots, the above procedure provides only one solution of (9). Adapting again the methods used for continuous variables (see for instance [5]) to the discrete case, a second linearly independent solution of (9) can be shown to be given by

$$\Phi_2(n) = \left[\frac{\partial \Phi(n,\sigma)}{\partial \sigma} \right]_{\sigma=0}. \tag{15}$$

In the above equation, $\Phi(n,\sigma)$ has the form (11), with $a_k(\sigma)$ given by (12). In order to actually evaluate $\Phi_2(n)$, we need to take the derivatives with respect to σ of $a_k(\sigma)$, and of $n^{(k+\sigma)}$, at $\sigma = 0$. It is easy to show, using (12), that

$$\frac{d}{d\sigma} a_k(\sigma) \mid_{\sigma=0} = -2a_k(0) H_k, \tag{16}$$

where we have defined

$$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}. \tag{17}$$

H_k is (up to a constant) a discrete version of the logarithmic function. To find $\frac{d}{d\sigma} n^{(k+\sigma)}$, we have to extend the definition of $n^{(k+\sigma)}$ to arbitrary real powers. Having in mind the representation of the factorial by Gamma functions, $n! = \Gamma(n) = \int_0^\infty dt e^{-t} t^n$, we define $n^{(k+\sigma)} = \frac{\Gamma(n)}{\Gamma(n-k-\sigma)}$. In the end, one obtains, dropping the part of the result which is proportional to $\Phi_1(n)$,

$$\Phi_2(n) = \sum_{k=0}^{n-1} \frac{(-\lambda)^k}{(k!)^2} n^{(k)} (H_{n-k} - 2H_k). \tag{18}$$

One can check explicitely that $\Phi_1(n)$ and $\Phi_2(n)$ satisfy (9).

We now consider the $n \to \infty$ limit, in order to see how the precedent solutions behave at distances r much larger than the noncommutativity scale $\sqrt{\theta}$. (This can also be seen as a small θ limit.) Using $\lambda = \theta\omega^2/2$ and $n = \frac{r^2}{2\theta} \to \infty$, $\Phi_1(n)$ becomes, as a function of r, the zero-order Bessel function of first type:

$$\Phi_1(n) \stackrel{n\to\infty}{\to} f_1(r) = \sum_{k=0}^{\infty} \frac{(-1)^k(\omega r)^{2k}}{(k!)^2 2^{2k}} = J_0(\omega r) \stackrel{r\to\infty}{\sim} \sqrt{\frac{2}{\pi\omega r}} cos(\omega r - \pi/4). \tag{19}$$

We see that $f_1(r)$ is independent of θ. This is not the case for the function of r one would obtain at finite n, which diverges as $\theta \to 0$. We will encounter only Bessel functions of zero order in what follows, since the angular dependence of Φ_n is lost, due to (2).

Similarly, Φ_2 becomes

$$\Phi_2(n) \to f_2(r) = \sum_{k=0}^{\infty} \frac{(-1)^k(\omega r)^{2k}}{(k!)^2 2^{2k}}[2ln(\omega r) - 2H_k + \gamma - ln(2\theta\omega^2)]. \tag{20}$$

γ is the Euler-Mascheroni constant, $\gamma = \lim_{k=\infty}(H_k - lnk) \simeq 0.5772$. Thus $f_2(r)$ still depends on θ, via a logarithmic term, which renders the $\theta \to 0$ limit singular. Using the series expansion of the Bessel function of second kind (Neumann function) [5], $Y_0(\omega r) \stackrel{r\to\infty}{\sim} \sqrt{\frac{2}{\pi\omega r}} sin(\omega r - \pi/4)$,

$$Y_0(\omega r) = \frac{2}{\pi}\left(\sum_{k=0}^{\infty} \frac{(-1)^{k+1}(\omega r)^{2k}}{(k!)^2 2^{2k}} H_k + (\gamma + ln(\omega r/2))J_0(\omega r)\right) \tag{21}$$

one sees that $f_2(r)$ is a linear combination of Bessel functions of first and second kind: $f_2(r)/\pi = Y_0(\omega r) + (\gamma + ln(2\theta\omega^2))J_0(\omega r)$.

Hence, the $n \to \infty$ limits of the solutions $\Phi_1(n)$ and $\Phi_2(n)$ obey the Bessel equation. This is in agreement with the $n = \frac{r^2}{2} \to \infty$ limit of the difference operator entering equation (9),

$$2(n\Delta^2\Phi_{n-1} + \Delta\Phi_n) \stackrel{n\to\infty}{\to} 2(n\frac{d^2}{dn^2} + \frac{d}{dn})\Phi(n) \stackrel{n=\frac{r^2}{2}}{=} (\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr})\Phi(r). \tag{22}$$

Thus, at $r >> \sqrt{\theta}$, noncommutative radial waves behave like ordinary, comutative ones. This allows us to find, via a 'correspondence principle', which combinations of $\Phi_1(n)$ and $\Phi_2(n)$ correspond to stationary, respectively traveling noncommutative waves: In the commutative case, it is well known that standing waves are described by the $J_0(r)$ function, whereas radially expanding ones by the first Hankel function $H_0^{(1)}(r) = J_0(r) + iY_0(r)$. Hence, the linear combination of $\Phi_1(n)$ and $\Phi_2(n)$ which at $n \to \infty$ tends to $J_0(\omega r)$ will describe standing noncommutative waves. This is obviously $\Phi_1(n)$, which consequently solves finite area boundary value problems with radial symmetry, describing standing oscillations. On the other hand, the function which tends to $H_0(\omega r)$ as $r \to \infty$, namely

$$\Phi_3(n) = \Phi_1(n) + \frac{i}{\pi}\left(\Phi_2(n) + [\gamma + ln(\theta\omega^2/2)]\Phi_1(n)\right), \tag{23}$$

represents a radial noncommutative wave propagating outwards. Now, any solution $\Phi(n)$ of (9) can be written as a linear superposition of $\Phi_1(n)$ and $\Phi_3(n)$, with coefficients determined by the boundary conditions one wishes to impose. It is understood that all the above solutions are multiplied by a dimensionfull, otherwise arbitrary, constant; the same will apply for sources.

It is worth noting that, in sharp contrast to the commutative case, in which Hankel and Neumann functions are singular at the origin, the functions $\Phi_{2,3}$ are nowhere singular (except when $\theta = 0$). This may suggest that, although not finite in quantum perturbation theory, fields defined over noncommutative spaces may not display *classical* divergences. This happens simply because the sources are not localized (also, one has no access to the origin: $r/\sqrt{\theta} = \sqrt{2n+1} \geq 1$). In order to support such a claim, one should include sources in the calculation, i.e, solve the inhomogeneous version of equation (9). The general problem is left for future work. Here we just outline an heuristic argument, for an hypothetical source j placed at $n = 0$, oscillating in time with frequency ω. That means equation (9) and its solutions are still valid for $n \geq 1$, whereas at $n = 0$, one gets $\Delta\Phi(0) + \lambda\Phi(0) = \Phi(1) - (1-\lambda)\Phi(0) = j$. Since $\Phi(1)$ is finite, cf. Eqs. (14) and (18), $\Phi(n=0)$ is finite too, even at the location of the source. The notable exception is $\lambda = \frac{\theta\omega^2}{2} = 1$, when $\Phi(0)$ diverges. The corresponding value, $\omega = \sqrt{\frac{2}{\theta}}$, happens to be the same one which cancels the $\theta-$dependent logarithmic term in equations (20) and (23).

Let us conclude with a summary of what we have shown:

- On the noncommutative plane defined by $[x^1, x^2] = i\theta$, radial waves propagate on a discrete space, given by the eigenvalues $r = \sqrt{2n+1}, n = 0, 1, 2, \ldots$ of the radius operator. One has no access to the origin ($r = 0$), as one would expect. The amplitude of the waves is given by a finite series, whose number of terms depends on the location at which the field amplitude is calculated: at radius $r = \sqrt{2n+1}\sqrt{\theta}$, one has $n+1$ terms in the series.

- In the large radius limit, $r >> \sqrt{\theta}$, or $n \to \infty$, the amplitudes become Bessel-type functions, consequently the waves behave like commutative ones.

- At small radius, if $\theta \neq 0$, there are no signs of singularities appearing, with the exception of a particular wave mode. This deserves to be further explored, including sources systematically into the calculation.

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